ENTROPY LOCKING

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ABSTRACT. We give a simple proof that in certain one-parameter families of piecewise continuous piecewise linear interval maps with two laps, topological entropy stays constant as the parameter varies.

1. INTRODUCTION

While continuous interval maps have been thoroughly investigated, the theory of piecewise continuous maps (except piecewise increasing ones) still presents a lot of open questions. Among discontinuous interval maps, the ones with two pieces of continuity/monotonicity (*laps*) are the simplest ones, so consider such maps, increasing on one lap and decreasing on the other one. To make them even simpler, let us assume that on each lap the map is affine, and in the interior of the lap on which the map is decreasing there is a fixed point. Using an affine conjugacy we can bring such a map to a form where the discontinuity occurs at 0 and the right limit at 0 of the value is 1. The formula will be then

$$T_{\lambda,\mu,b}(x) = \begin{cases} 1 + \lambda x + b & \text{if} \quad x \le 0, \\ 1 - \mu x & \text{if} \quad x \ge 0, \end{cases} \tag{1}$$

where $\lambda, \mu > 0$; see Figure 1.

Observe that unless $b = 0$, our maps take two values at 0. However, this will not create any problems. One should think about the point 0 as sometimes being two points, 0_- and 0_+. Thus, we will write $T_{\lambda,\mu,b}(0_-)$ for $\lim_{x \nearrow 0} T_{\lambda,\mu,b}(x)$ and $T_{\lambda,\mu,b}(0_+)$ for $\lim_{x \searrow 0} T_{\lambda,\mu,b}(x)$. Moreover, if λ and μ are fixed, then we will simply write T_b for $T_{\lambda,\mu,b}$.

Define $y_b = \max\{T_b(0_-), T_b(0_+)\}$ and $x_b = T_b(y_b)$. We want to consider our map on a compact interval instead of the whole real line. The natural candidate for this interval is $[x_b, y_b]$. If this interval is invariant for T_b, then it is the smallest invariant interval. If this interval is not invariant, then the trajectory of x_b escapes to $-\infty$, and there is no invariant interval. The necessary and sufficient condition for this interval to be invariant is $T_b(x_b) \in [x_b, y_b]$. Since always $T_b(x_b) < y_b$, our condition becomes

$$T_b(x_b) \ge x_b. \tag{2}$$

While we could translate (2) to inequalities in λ, μ and b, we would never use them in that form.

We also want the map to be (eventually) piecewise expanding, so we assume that

$$\lambda \ge 1 \quad \text{and} \quad \mu > 1. \tag{3}$$

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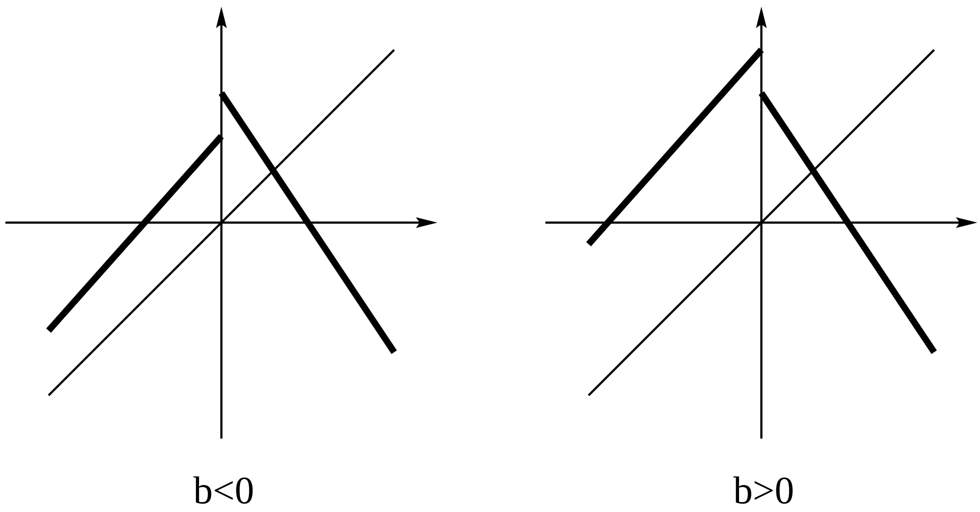

FIGURE 1. The maps $T_{\lambda,\mu,b}$.

However, if in both (2) and (3) we have equalities, then the map on the left lap is the identity. This is a highly degenerate case, so we will assume that

$$\text{if} \quad T_b(x_b) = x_b \quad \text{then} \quad \lambda > 1. \tag{4}$$

Throughout most of the paper we will consider maps $T_b = T_{\lambda,\mu,b}$ satisfying (2), (3) and (4). We will denote the family of those maps by $\mathcal{T}$.

The map T_b has a fixed point

$$z = \frac{1}{1+\mu}$$

on the right lap. Note that its position does not depend on b, so we do not need a subscript b here.

For a piecewise continuous piecewise monotone map f (with the finite number of laps), the usual definition of its topological entropy is

$$h_{\text{top}}(f) = \lim_{n\to\infty} \frac{1}{n} \log c_n, \tag{5}$$

where c_n is the number of laps of f^n. In [8] it is shown that this agrees with the standard Bowen's definition of topological entropy.

In 2013, V. Botella-Soler, J. A. Oteo, J. Ros and P. Glendinning [5] noticed that for certain values of λ and μ both Lyapunov exponent and topological entropy of $T_{\lambda,\mu,b}$ remain constant as b varies in some interval of values close to 0, although the kneading sequence varies. In 2014, H. Bruin, C. Carminati, S. Marmi and A. Profeti [3] explained this phenomenon by observing that it is caused by *matching with index zero*, where for some $k > 0$ we have $T_b^k(0_-) = T_b^k(0_+)$. In fact, in their definition of matching there is also a similar condition on the derivatives, but we do not need it here (although it is satisfied).

In this paper we are giving a simple explanation of the matching with index zero phenomenon and of the fact that the topological entropy remains constant (we call this phenomenon *entropy locking*).

2. Kneading theory

Kneading theory is a standard tool for studying maps of the interval. We will use the notation of [6]. For $x \in [x_b, y_b]$ we define its itinerary $I(x)$ to be the sequence $I_0(x)I_1(x)I_2(x)\dots$, where

$$I_0(x) = \begin{cases} R & \text{if } x > 0, \\ C & \text{if } x = 0, \\ L & \text{if } x < 0, \end{cases} \tag{6}$$

and $I_j(x) = I_0(f^j(x))$. We adopt the convention that the itinerary terminates if $I_j(x) = C$ for some j. An arbitrary sequence $\underline{A}$ of Rs, Ls, and Cs will be called *admissible* if it is either an infinite sequence of Rs and Ls, or a finite (possibly empty) sequence of Rs and Ls followed by a C. Note that all itineraries are admissible.

The length of a sequence $\underline{A}$ of R's and L's will be denoted by $|\underline{A}|$, and $\underline{A}$ will be called *finite* if $|\underline{A}| < \infty$. The *parity* of a finite sequence $\underline{A}$ of R's and L's describes the number of R's in the sequence. The sequence $\underline{A}$ is *even* if the number of Rs is even and *odd* if the number of Rs is odd. These sequences can be ordered in the following manner.

Definition 2.1. First we have a natural ordering $L < C < R$. Suppose that $\underline{A} = A_0A_1\dots$ and $\underline{B} = B_0B_1\dots$ are admissible sequences such that $\underline{A} \neq \underline{B}$. Let n be the first index such that $A_n \neq B_n$. If the finite sequence

$$A_0A_1\dots A_{n-1} = B_0B_1\dots B_{n-1}$$

is even and $A_n < B_n$, then $\underline{A} < \underline{B}$. If this finite sequence is odd and $A_n < B_n$, then $\underline{A} > \underline{B}$.

This is known as the *parity-lexicographical ordering* and it agrees with the ordering of the points on the interval.

Proposition 2.2. *For a map $T_b \in \mathcal{T}$, $I(x) < I(y)$ if and only if $x < y$.*

We will not prove this proposition here, since its proof is practically identical to the proof for continuous maps (see, e.g., [6]). The only detail that is different, is that we can use the strict inequalities on both sides of the equivalence. This follows from the fact that our maps have iterates that are piecewise expanding, so different points have different itineraries. Let us state it as a lemma.

Lemma 2.3. *For a map $T_b \in \mathcal{T}$, there is n such that T_b^n is expanding on each lap.*

Proof. If $\lambda > 1$, then T_b itself is expanding on each lap. Assume that $\lambda = 1$. Then, by (2) and (4), $T_b(x_b) - x_b > 0$, and for each $x \in [x_b, 0)$ we have $T_b(x) - x = T_b(x_b) - x_b$. This means that at least one of the points $T_b^i(x)$, $0 \le i \le n$, belongs to the right lap of T_b, provided $n > |x_b|/(T_b(x_b) - x_b)$. Therefore, for such n the map T_b^n is expanding with the constant at least μ on each lap. □

We define the *left and right kneading sequences* of T_b to be $K_-(T_b) = I(T_b(0_-))$ and $K_+(T_b) = I(T_b(0_+))$.

3. Matching

We are interested in the conditions under which $T_b^k(0_-)$ and $T_b^k(0_+)$ coincide for some k. We start with a simple geometric lemma.

Lemma 3.1. *Let f be a map conjugated to $T_{\lambda,\mu,0} \in \mathcal{T}$ via an orientation preserving affine map. Let c be the turning point of f and let $x < c < y$. Then $f(x) = f(y)$ if and only if*

$$\frac{x-c}{c-y} = \frac{\mu}{\lambda}. \tag{7}$$

Proof. Assume that (7) is satisfied. Then

$$f(x) - f(c) = \lambda(x-c) = \mu(c-y) = f(y) - f(c),$$

and therefore $f(x) = f(y)$.

Now assume that $f(x) = f(y)$. Then

$$\lambda(x-c) = f(x) - f(c) = f(y) - f(c) = \mu(c-y),$$

and (7) follows. □

Now we can prove the main result of this section. In the proof we will be using the notation $\langle x, y\rangle$ for $[x, y]$ if $x < y$ and $[y, x]$ if $y < x$.

Theorem 3.2. *Let $T_b = T_{\lambda,\mu,b} \in \mathcal{T}$, and let $\underline{A}$ be a finite (possibly empty) sequence of symbols R and L. Set $n = |RL\underline{A}C|$. Assume that $K_-(T_b) = RL\underline{A}R\dots$ and $K_+(T_b) = RL\underline{A}L\dots$. Then $K(T_0) = RL\underline{A}C$ if and only if $T_b^{n+1}(0_-) = T_b^{n+1}(0_+)$.*

Proof. We use the ideas from the Euclidean geometry. We consider the graph of T_b, then draw some additional lines, identify similar figures and use proportions.

Thus, consider the graph of T_b. It consists of two branches. From the assumptions on the kneading sequences it follows that $b \neq 0$. If $b < 0$, then the left branch ends lower than the right branch; if $b > 0$ then the right branch ends lower than the right one. Extend the lower branch until it crosses the higher one (see Figure 2). This happens at the point $(c, T_b(c))$, where $1 + \lambda c + b = 1 - \mu c$, so

$$c = \frac{-b}{\mu+\lambda}. \tag{8}$$

Now we define a continuous map f of $[x_b, y_b]$ to itself by

$$f(x) = \begin{cases} 1 + \lambda x + b & \text{if} \quad x \le c, \\ 1 - \mu x & \text{if} \quad x \ge c. \end{cases}$$

We claim that $f^i(c) \notin \langle 0, c\rangle$ for $i = 1, 2, \dots, n-1$. Indeed, suppose that $f^i(c) \in \langle 0, c\rangle$ for some $i \in [1, n-1]$ and $f^k(c) \notin \langle 0, c\rangle$ for all $k \in [1, i-1]$. Then $f^k(c) = T_b^k(c)$ for $k \in [1, i]$. Set $U = \langle T_b(0_-), T_b(0_+)\rangle$, and note that $T_b(c) \in U$.

Since both $K_-(T_b)$ and $K_+(T_b)$ begin with RL, the interval U lies to the right of the fixed point z, while 0 and c are to the left of z. Therefore $i \ge 2$.

We have

$$T_b(0_+) - T_b(c) = -\mu\left(0 - \frac{-b}{\mu+\lambda}\right) = \frac{-\mu b}{\mu+\lambda} \tag{9}$$

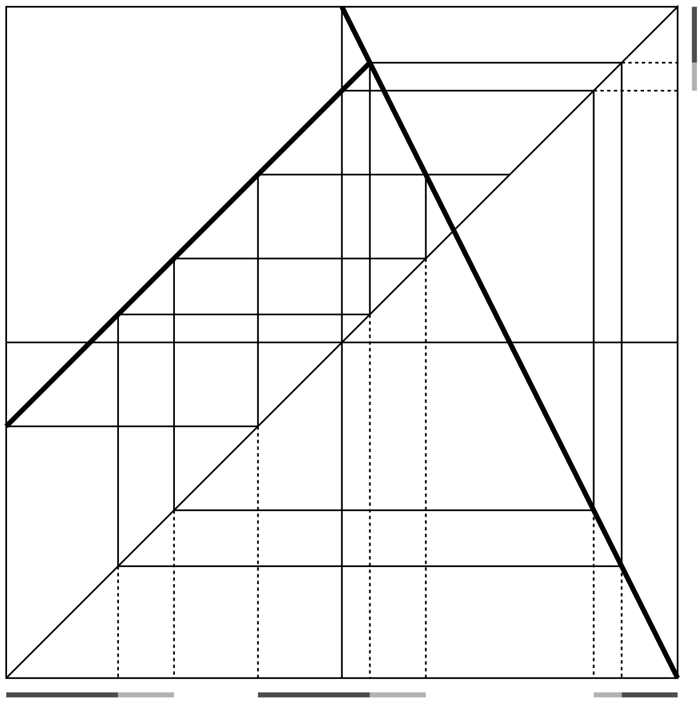

FIGURE 2. The proof of Theorem 3.2. The proportion of the lengths of the light gray and dark gray intervals stays λ/μ.

and

$$T_b(c) - T_b(0_-) = \lambda\left(\frac{-b}{\mu+\lambda} - 0\right) = \frac{-\lambda b}{\mu+\lambda}. \tag{10}$$

Since $K_+(T_b)$ and $K_-(T_b)$ agree on the first $n-1$ places, then $0 \notin T_b^k(U)$ for $k \leq i$. Therefore, T_b^i is affine on U. Thus, we get

$$|T_b^i(0_+) - T_b^i(c)| \geq \frac{\mu^2|b|}{\mu+\lambda} > |c|$$

and

$$|T_b^i(c) - T_b^i(0_-)| \geq \frac{\lambda\mu|b|}{\mu+\lambda} > |c|,$$

where we get the final inequality because $\mu > 1$ and $\lambda \geq 1$. Thus, $K_+(T_b)$ and $K_-(T_b)$ disagree on the $i-1$st index, which is a contradiction. This proves that $f^i(c) \notin \langle 0, c\rangle$ for $i = 1, 2, \ldots, n-1$.

It follows from the assumption on kneading sequences that $n \geq 3$, and therefore $f^2(c) \notin \langle 0, c\rangle$, so $f^2(c) < c$. This implies that the interval $[f^2(c), f(c)]$ is invariant under f. Since f has the same slopes as T_0, then $f|_{[f^2(c),f(c)]}$ is conjugate to T_0 via an orientation preserving affine map. Moreover, it follows from our claim that $f^i(c) = T_b^i(c)$ for all $i \in [0, n]$. Because of the assumptions on kneading sequences, none of the intervals $T_b^i(U)$, $0 < i < n-1$, contains 0. Therefore the map T_b^{n-1} is affine on U.

From this and from the formulas (9) and (10) it follows that

$$\frac{T_b^n(0_+) - T_b^n(c)}{T_b^n(c) - T_b^n(0_-)} = \frac{\mu}{\lambda}. \tag{11}$$

Assume that $K(T_0) = RL\underline{A}C$. Then $f^n(c) = c$. We have $|T_b^n(0_-) - c| > |c|$ and $|T_b^n(0_+) - c| > |c|$, and thus, $T_b^n(0_-)$ and $T_b^n(0_+)$ are not contained in the interval $\langle 0, c \rangle$. Hence, $T_b^{n+1}(0_+) = f(T_b^n(0_+))$ and $T_b^{n+1}(0_-) = f(T_b^n(0_-))$, so from (11) and Lemma 3.1 we get $T_b^{n+1}(0_-) = T_b^{n+1}(0_+)$.

Assume now that $T_b^{n+1}(0_-) = T_b^{n+1}(0_+)$. Then $T_b^{n+1}(0_-) \le \min(T_b(0_-), T_b(0_+)) < T_b(c)$, so again $T_b^n(0_-)$ and $T_b^n(0_+)$ are not contained in the interval $\langle 0, c \rangle$. By (11) and Lemma 3.1 we get $f^n(c) = c$. Since for $i < n$ the point $f^i(c)$ is between $T_b^i(0_-)$ and $T_b^i(0_+)$ and both $K_-(T_b)$ and $K_+(T_b)$ begin with $RL\underline{A}$, and moreover, $f^i(c) \notin \langle 0, c \rangle$ for $i = 1, 2, \dots, n-1$, we see that the kneading sequence of f also begins with $RL\underline{A}$. Since $f^n(c) = c$, the next symbol is C. The maps T_0 and f are conjugate, so their have the same kneading sequences. Therefore, $K(T_0) = RL\underline{A}C$. □

Remark 3.3. Suppose that the assumptions of Theorem 3.2 are satisfied. If $b < 0$ then $T_b(0_-) < T_b(0_+)$, so $K_-(T_b) < K_+(T_b)$. This implies that $\underline{A}$ is even. Similarly, if $b > 0$ then $\underline{A}$ is odd.

We can prove a kind of converse to the above remark.

Proposition 3.4. *Fix parameters $\lambda \ge 1$, $\mu > 1$, such that $K(T_0) = RL\underline{A}C$. Then if if $\underline{A}$ is even (respectively, odd), there exists $\varepsilon > 0$ such that if $b \in (-\varepsilon, 0)$ (respectively, $b \in (0, \varepsilon)$) then the assumptions of Theorem 3.2 are satisfied, and thus, $T_b^{n+1}(0_-) = T_b^{n+1}(0_+)$.*

Proof. If $|b|$ is sufficiently small, then both $K_-(T_b)$ and $K_+(T_b)$ begin with $RL\underline{A}$. Thus, we have to show that the next symbol is R for $K_-(T_b)$ and L for $K_+(T_b)$. By making the construction from the proof of Theorem 3.2, we see that $T^n(0_-)$ and $T^n(0_+)$ are on the opposite sides of c. Moreover, both $|T^n(0_-) - c|$ and $|T^n(0_+) - c|$ are larger than $|c|$, so the nth terms of $K_-(T_b)$ and $K_+(T_b)$ are distinct. Taking into account the order in the set of itineraries (as in Remark 3.3), we get the assertion of the proposition. □

4. Topological entropy

Entropy locking refers to intervals of the parameter b where topological entropy of T_b remains constant. It turns out that the intervals of parameter b satisfying Theorem 3.2 are intervals with entropy locking.

We need some estimates of the topological entropy for piecewise continuous piecewise monotone interval maps (when using this term, we always assume that the number of pieces is finite). They are known, but they are difficult to find in the literature. Since the proofs are simple, we provide them here.

For a piecewise continuous piecewise monotone interval map f we will say that α is an *anti-Lipschitz constant* if for every x, y from the same lap we have $|f(x) - f(y)| \ge \alpha|x - y|$. In particular, a map with an anti-Lipschitz constant larger than 1 is piecewise expanding.

An *s-horseshoe* for f is an interval J and a partition $D = \{J_1, \dots, J_s\}$ of J into s subintervals such that $J \subset f(J_i)$ and f is continuous and monotone on each J_i. The following theorem was proved in [8].

Theorem 4.1. *If f is a piecewise continuous piecewise monotone interval map, then for every $\varepsilon > 0$ there exist n and s, such that f^n has an s-horseshoe and $(1/n)\log s > h_{\mathrm{top}}(f) - \varepsilon$.*

Now we can prove the promised estimates.

Theorem 4.2. *If f is a piecewise continuous piecewise monotone interval map with an anti-Lipschitz constant α and a Lipschitz constant β, then $\log\alpha \le h_{\mathrm{top}}(f) \le \log\beta$.*

Proof. We use formula (5). If the interval on which f is acting has length γ, then the length of each lap of f^n is not larger than γ/α^n. Therefore $c_n \ge \alpha^n$, and thus, $h_{\mathrm{top}}(f) \ge \log\alpha$.

Take $\varepsilon > 0$. By Theorem 4.1, there exist n and s, such that f^n has an s-horseshoe and $(1/n)\log s > h_{\mathrm{top}}(f) - \varepsilon$. Let an interval J and a partition $D = \{J_1, \dots, J_s\}$ be this horseshoe. Then the length of each J_i is at least the length of J divided by β^n. Therefore, $s \le \beta^n$, and hence, $\log\beta > h_{\mathrm{top}}(f) - \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we get $h_{\mathrm{top}}(f) \le \log\beta$. □

From this theorem and Lemma 2.3, we get immediately the following corollary.

Corollary 4.3. *All maps from $\mathcal{T}$ have strictly positive topological entropy.*

Any map $T_{\lambda,\lambda,b} \in \mathcal{T}$ has both anti-Lipschitz and Lipschitz constants equal to λ. Therefore we get immediately another corollary to Theorem 4.2.

Corollary 4.4. *If $T_{\lambda,\lambda,b} \in \mathcal{T}$, then its topological entropy is $\log\lambda$.*

Now we are ready to prove the main result of this section. We will refer to piecewise continuous piecewise affine interval maps with the absolute value of the derivative constant, as *maps of constant slope.* In $\mathcal{T}$, these are maps of the form $T_{\lambda,\lambda,b}$.

We will be using often a certain long assumption, so it makes sense to give it a short name.

Definition 4.5. We will say that T_b satisfies the *kneading assumption* if $T_b = T_{\lambda,\mu,b} \in \mathcal{T}$ and there exists a finite (possibly empty) sequence $\underline{A}$ of symbols R and L, such that $K(T_0) = RL\underline{A}C$, $K_-(T_b) = RL\underline{A}R\dots$, and $K_+(T_b) = RL\underline{A}L\dots$.

Theorem 4.6. *Assume that T_b satisfies the kneading assumption and is topologically conjugate to a map of constant slope. Then $h_{\mathrm{top}}(T_b) = h_{\mathrm{top}}(T_0)$.*

Proof. By the assumption, $T_{\lambda,\mu,b}$ is conjugate to $T_{\alpha,\alpha,d}$ for some α and d. By Corollary 4.4,

$$\log\alpha = h_{\mathrm{top}}(T_{\alpha,\alpha,d}) = h_{\mathrm{top}}(T_{\lambda,\mu,b}). \tag{12}$$

Set $n = |RL\underline{A}C|$. From Theorem 3.2 it follows that $T^{n+1}_{\lambda,\mu,b}(0_+) = T^{n+1}_{\lambda,\mu,b}(0_-)$. Hence, $T^{n+1}_{\alpha,\alpha,d}(0_+) = T^{n+1}_{\alpha,\alpha,d}(0_-)$. Since the kneading sequences are preserved by a conjugacy, the left and right kneading sequences of $T_{\alpha,\alpha,d}$ are $K_-(T_{\alpha,\alpha,d}) = RL\underline{A}R\dots$ and $K_+(T_{\alpha,\alpha,d}) = RL\underline{A}L\dots$, respectively. Thus, we can use Theorem 3.2 again, and we get $K(T_{\alpha,\alpha,0}) = RL\underline{A}C$. For unimodal maps the topological entropy is determined by the kneading sequence, and therefore

$$h_{\mathrm{top}}(T_{\lambda,\mu,0}) = h_{\mathrm{top}}(T_{\alpha,\alpha,0}). \tag{13}$$

By Corollary 4.4,

$$h_{\mathrm{top}}(T_{\alpha,\alpha,0}) = \log \alpha. \tag{14}$$

From (12), (13) and (14) we get $h_{\mathrm{top}}(T_{\lambda,\mu,0}) = h_{\mathrm{top}}(T_{\lambda,\mu,b})$. □

5. Transitivity

While Theorem 4.6 is quite strong, it contains an assumption that may be not easy to verify in concrete situations. Namely, we assume that T_b is topologically conjugate to a map of constant slope. In this section we will try to replace this assumption by weaker ones, which are easier to verify.

The first idea is to assume that T_b is topologically transitive. The following theorem can be found for instance in [2].

Theorem 5.1. *If f is a piecewise continuous piecewise monotone topologically transitive interval map with topological entropy $\log \beta > 0$, then it is topologically conjugate to a map of constant slope β.*

In view of this theorem and Corollary 4.3, we get the following corollary to Theorem 4.6.

Corollary 5.2. *Assume that T_b satisfies the kneading assumption and is topologically transitive. Then $h_{\mathrm{top}}(T_b) = h_{\mathrm{top}}(T_0)$.*

We will further improve this corollary, by replacing the assumption that T_b is topologically transitive by another assumption, which is maybe a little weaker, but easier to check. This assumption will be

$$T_{\lambda,\mu,0}(x_0) < z. \tag{15}$$

It can be easily written as an inequality on parameters

$$\lambda + \mu < \lambda\mu^2. \tag{16}$$

It is known that it is equivalent to $T_{\lambda,\mu,0}$ being totally transitive; however, we will not use this fact. We will say that $T_b = T_{\lambda,\mu,b}$ satisfies (15) if $T_0 = T_{\lambda,\mu,0}$ satisfies it.

Definition 5.3. The set $\mathcal{T}_{\mathrm{KAT}}$ is the set of all maps T_b satisfying both the kneading assumption and (15).

Lemma 5.4. *Assume that $T_b \in \mathcal{T}_{\mathrm{KAT}}$. Then*

$$T_b(1-\mu) \le 1. \tag{17}$$

Proof. If $b \le 0$, then $y_b = 1$, so (17) holds. Assume that $b > 0$. If $T_b(1-\mu) > 1$, then $K_+(T_b) = RLRL\dots$. By the kneading assumption, $K_-(T_b) = RLR\dots$. We have $T_b(x_b) = 1 + \lambda(1-\mu-\mu b) + b$ and $T_0(x_0) = 1 + \lambda(1-\mu)$. Since $b < \lambda\mu b$, we get $T_b(x_b) < T_0(x_0)$. By this and (15), $T_b(x_b) < z$, so the next term in $K_-(T_b)$ is R. Thus, by the kneading assumption, $K(T_0) = RLRC$. Then $1 - \mu(1+\lambda-\lambda\mu) = T_0^4(0) = 0$, so $\lambda = 1/\mu < 1$, a contradiction. Thus, (17) holds. □

Lemma 5.5. *Assume that $T_b \in \mathcal{T}_{\mathrm{KAT}}$. Let U be an interval containing z. Then*

$$\bigcup_{i=0}^{\infty} T_b^i(U) = [x_b, y_b].$$

Proof. Suppose first that $b \le 0$. Then $[x_b, y_b] = [1-\mu, 1]$. Since the interval U contains z, then all sets $T_b^i(U)$ must contain z as well. Moreover, $\mu > 1$, so the length of $T_b^i(U)$ is expanding exponentially with i until we reach an m such that $T_b^m(U)$ contains $[z, 1]$. Therefore $T_b([z,1]) = [1-\mu, z] \subset T_b^{m+1}(U)$. Hence, $T_b^m(U) \cup T_b^{m+1}(U) = [x_b, y_b]$.

Now assume that $b > 0$. By Lemma 5.4, (17) holds. As in the case $b \le 0$, we get $T_b^m(U) \cup T_b^{m+1}(U) \supset [1-\mu, 1]$ for some m. Since $T_b(1-\mu) \le 1$, the interval $T_b([1-\mu, 0])$ contains $[1, y_b]$. Since $T_b([1, y_b]) = [x_b, 1-\mu]$, we get $T_b^m(U) \cup T_b^{m+1}(U) \cup T_b^{m+2}(U) \cup T_b^{m+3}(U) = [x_b, y_b]$. $\square$

Theorem 5.6. *Assume that $T_b \in \mathcal{T}_{\mathrm{KAT}}$. Then T_b is topologically transitive.*

Proof. Let U be an open subinterval of $[x_b, y_b]$. We will show that $V = \bigcup_{i=0}^{\infty} T_b^i(U)$ is dense in $[x_b, y_b]$. Since $\mu > 1$, the length of $T_b^n(U)$ increases exponentially with n. Thus, there exists k such that $0 \in T_b^k(U)$. Therefore, V contains an interval containing 0. Let W be the largest such interval contained in V. We can write $W = W_L \cup W_R$, where $W_L = \{x \in W : x \le 0\}$ and $W_R = \{x \in W : x \ge 0\}$. Since V is invariant and $\mu > 1$, then, by the same reason as for U, it must happen that 0 belongs to the interior of $T_b^m(W_L)$ and $T_b^n(W_R)$ for some positive integers m and n. If m and n are minimal such integers, then $T_b^m(W_L)$ and $T_b^n(W_R)$ are intervals, and therefore they are contained in W.

Suppose that V is not dense. We claim that then $m \ge 2$ and $n \ge 2$. In view of Lemma 5.5, in order to prove the claim, it is enough to show that if m or n is 1, then $z \in W$.

Assume first that $b < 0$. If $m = 1$, then $T_b(W_L) \subset W$ and in particular $T_b(0_-) \in W$. Therefore the interval $[0, T_b(0_-)]$ is contained in W. We claim that $T_b(0_-) \ge z$. Indeed, if $T_b(0_-) < z$, then $K_-(T_b)$ starts with RR, which is impossible by the kneading assumption, and this proves the claim. Therefore, $z \in [0, T_b(0_-)]$. If $n = 1$, then $T_b(W_R) \subset W$, and in particular $T_b(0_+) = 1 \in W$. Thus, $z \in [0,1] \subset W$.

Now assume that $b > 0$. If $m = 1$, then $T_b(W_L) \subset W$ and in particular $T_b(0_-) = 1 + b \in W$. Thus, $z \in [0, 1+b] \subset W$. If n=1, then $T_b(W_R) \subset W$ and it follows that $z \in [0,1] \subset W$. This completes the proof of the claim.

By our choice of m and n, T_b^m is affine on W_L and T_b^n is affine on W_R. Additionally, since $T_b(0_-) > 0$, $T_b(0_+) > 0$ and $m, n \ge 2$, we have $I(x) = LR\dots$ for every $x \in W_L$ and $I(x) = RR\dots$ for every $x \in W_R$.In such a way, we get lower bounds on the lengths of $T_b^m(W_L)$ and $T_b^n(W_R)$:

$$\begin{aligned} \lambda\mu|W_L| &\le |T_b^m(W_L)|, \\ \mu^2|W_R| &\le |T_b^n(W_R)|. \end{aligned} \tag{18}$$

We also know that $T_b^m(W_L) \subset W$ and $T_b^n(W_R) \subset W$, so from (18) we get

$$\begin{aligned} \lambda\mu|W_L| &\le |W|, \\ \mu^2|W_R| &\le |W|. \end{aligned} \tag{19}$$

We add the first inequality in (19) multiplied by by μ to the second one multiplied by λ, and taking into account that $|W_L| + |W_R| = |W|$, we get

$$\lambda\mu^2|W| \leq (\lambda + \mu)|W|,$$

which contradicts (16) (which, as we noticed, is equivalent to (15)). This completes the proof. $\square$

Now from Corollary 5.2 and Theorem 5.5 we get an improved corollary.

Corollary 5.7. *Assume that* $T_b \in \mathcal{T}_{\mathrm{KAT}}$. *Then* $h_{\mathrm{top}}(T_0) = h_{\mathrm{top}}(T_b)$.

6. Beyond transitivity

Theorem 5.6 gives sufficient conditions for transitivity of $T_b = T_{\lambda,\mu,b}$. The assumption of this theorem is that $T_b \in \mathcal{T}_{\mathrm{KAT}}$, that is, that T_b satisfies the kneading assumption and satisfies 15. The examples in this section will show that both assumptions are essential.

First, we establish a necessary condition for transitivity.

Lemma 6.1. *Suppose* $T_b \in \mathcal{T}$. *If* $z \notin T_b([x_b, 0])$, *then* T_b *is not transitive.*

Proof. Let ε be sufficiently small so that $(z-\varepsilon, z+\varepsilon) \cap T_b([x_b, 0]) = \emptyset$. Since $\mu > 1$, z is repelling, and therefore $T_b^{-1}((z-\varepsilon, z+\varepsilon)) \subset (z-\varepsilon, z+\varepsilon)$. Hence, if V is an open interval such that $(z-\varepsilon, z+\varepsilon) \cap V = \emptyset$, then $T_b^n(V) \cap (z-\varepsilon, z+\varepsilon) = \emptyset$ for all n. Thus, T_b is not transitive. $\square$

Example 6.2. Set $\lambda = 1$ and find μ such that the kneading sequence of $T_0 = T_{\lambda,\mu,0}$ is $RLRRRC$. Elementary computations show that μ is the real solution of the equation $\mu^3 - \mu^2 - 1 = 0$ ($\mu \approx 1.46557$). We can deduce from the kneading sequence that $T_0(x_0) > z$, so T_0 does not satisfy (15). Moreover, $T_b(x_b) > z$ for sufficiently small b. It follows from Proposition 3.4 that T_b satisfies the kneading assumption for sufficiently small $b > 0$. Hence, for $b > 0$ sufficiently small, T_b satisfies the kneading assumption, but not (15), and is not transitive.

Example 6.3. Set $\lambda = 1$ and $\mu = 2$. Then $K(T_0) = RLC$ and $T_0(x_0) < z$. Therefore, T_b satisfies (15) for any b. However, for $b = -\frac{3}{4}$ we have $T_b(0_-) < z$, so by Lemma 6.1, T_b is not transitive. In particular, it cannot satisfy the kneading assumption.

We will show that also the topological entropies of T_0 and T_b are different. Both maps are Markov. For T_0, the Markov partition consists of two intervals, and its Markov graph is as in Figure 3. Therefore, the topological entropy of T_0 is the logarithm of the positive solution of the equation $x^2 - x - 1 = 0$, that is, the logarithm of the golden ratio $\phi = \frac{1+\sqrt{5}}{2} \approx 1.618$.

$$\begin{tikzcd} I_1 \arrow[r, shift left] & I_2 \arrow[l, shift left] \arrow[loop right] \end{tikzcd}$$

Figure 3. Markov graph for T_0.

For the map T_b, we have $T_b^6(0_+) = 0$ and $T_b^3(0_-) = 0$. A Markov partition $\mathcal{P}$ of $[x_b, y_b] = [-1, 1]$ is given by the orbits of 0_+ and 0_-. We shall denote the intervals of this partition by J_i, as illustrated in Figure 4.

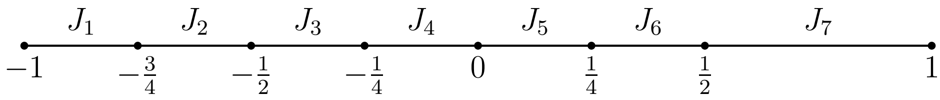

FIGURE 4. Markov partition $\mathcal{P}$ for T_b.

The Markov graph for T_b with the partition $\mathcal{P}$ is presented in Figure 5. One can find easily its entropy using the rome method (see [4] or [1]). It is equal to the logarithm of the positive solution of the equation $x^6 - x^3 - x^2 - x - 1 = 0$, that is, approximately $\log 1.3803$. Hence, $h_{\mathrm{top}}(T_0) \neq h_{\mathrm{top}}(T_b)$ for $b = -3/4$. A reader, that does not believe in approximate values, can check that

$$x^6 - x^3 - x^2 - x - 1 = (x^4 + x^3 + 2x^2 + 2x + 3)(x^2 - x - 1) + (4x + 2),$$

so $\phi^6 - \phi^3 - \phi^2 - \phi - 1 = 4\phi + 2 > 0$.

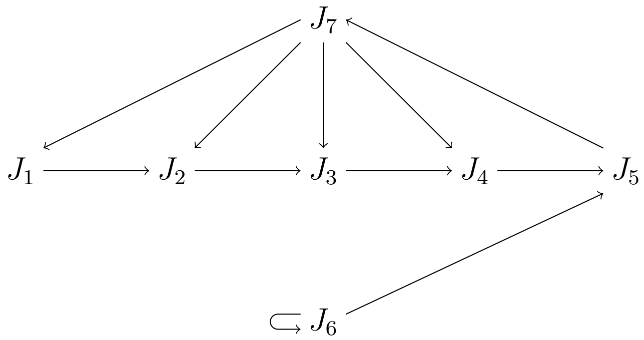

FIGURE 5. Markov graph associated to the partition $\mathcal{P}$.

Remember that the reason we started to consider transitivity of T_b was that we do not know any other simple way of verifying that T_b is conjugate to a map of constant slope. However, the maps $T_0 \in \mathcal{T}$ are known to be conjugate to maps of constant slope (this basically follows from [7] and [6], although it is not stated explicitly there). Thus, we can state the following conjecture.

Conjecture 6.4. *Every $T_b \in \mathcal{T}$ is topologically conjugate to a map of constant slope.*

If this conjecture is true, then by Theorem 4.6 every map $T_{\lambda,\mu,b} \in \mathcal{T}$ satisfying the kneading assumption would have the same topological entropy as $T_{\lambda,\mu,0}$.

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